

# On geometric conditions for reduction of the Moreau sweeping process to the Prandtl-Ishlinskii operator

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## Abstract

The sweeping process was proposed by J. J. Moreau as a general mathematical formalism for quasistatic processes in elastoplastic bodies. This formalism deals with connected Prandtl's elastic-ideal plastic springs, which can form a system with an arbitrarily complex topology. The model describes the complex relationship between stresses and elongations of the springs. On the other hand, the Prandtl-Ishlinskii model assumes a very simple connection of springs. This model results in an input-output operator, which has many good mathematical properties. It turns out that the sweeping processes can be reducible to the Prandtl-Ishlinskii operator even if the topology of the system of springs is complex. In this work, we analyze the conditions for such reducibility.

## 1 Introduction

Models of quasistatic elasto-plasticity date back to Prandtl's model of an elastic-ideal plastic element, which can be thought of as a cascade connection of an ideal Hook's spring and a Coulomb dry friction element. This simple model accounts for two important effects, saturation of stress with increasing deformation (strain) and hysteresis in the stress-strain relationship. Hysteresis is a manifestation of the fact that stress at a moment  $t$  is not a single-valued function of the concurrent deformation value, but rather a function of state of the elasto-plastic material, which depends on the history of variations of the deformation prior to the instant  $t$ . Two parameters of Prandtl's model are the stiffness  $a$  of the spring and the maximal spring force  $r$  (which equals the friction force in the sliding regime for quasistatic deformations).

In order to account for complex relationship between deformation and stress in real materials, Prandtl proposed to model the constitutive law of the material with a parallel connection of elastic-ideal plastic elements. A similar idea was developed by Ishlinskii who modeled individual fibers of wire ropes by Prandtl's elements. In the Prandtl-Ishlinskii phenomenological model, a finite or infinite set of Prandtl's elements (characterized by different values  $(a_i, r_i)$  of parameters of stiffness and maximal stress) are all subject to the same deformation  $\varepsilon(t)$ , and the total force (or stress)  $\sigma(t)$  is proportional to the sum of all spring forces. The operator that maps the time series  $\varepsilon(t)$  of the deformation (input) to the time series of stress  $\sigma(t)$  (output), given a set of initial stresses of all the springs (initial state), is known as the Prandtl-Ishlinskii (PI) operator in one-dimensional elasto-plasticity. Thanks to the set of good mathematical properties of this operator (see, for example, [1–7]), its equivalent counterparts have been used in

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several other disciplines including tribology (Maxwell-slip friction model), damage counting and fatigue estimation (the rain flow counting method), and, more recently, modeling constitutive laws of smart materials such as piezo-electric and magnetostrictive materials and shape memory alloys. One useful property, called the composition rule, is that a composition of PI operators is also a PI operator and, as a consequence, the inverse operator for a PI operator is another PI operator. Furthermore, a PI operator and its inverse admit an efficient analytic and numerical implementation. This property, in particular, underpins the design of compensation-based control schemes for microactuators and sensors, which use smart materials for energy conversion.

Another fact that facilitates modeling various constitutive laws with the PI operator, and is also central to this paper, is stated by the Representation Theorem, which allows one to determine whether a set of input-output data can be modeled by a PI operator and, moreover, equips one with an algorithm for identifying parameters of a PI operator from a simple measurement procedure. The Representation Theorem states that if (a) the input-output relationship between deformation and stress is *rate-independent*<sup>2</sup>; (b) hysteresis loops corresponding to periodic inputs are *closed*<sup>3</sup>; and, (c) every hysteresis loop is *centrally symmetric*, then  $\sigma(t) = (\mathcal{P}_\phi \varepsilon)(t)$ , where  $\mathcal{P}_\phi : C(0, T) \rightarrow C(0, T)$  is a PI operator. Furthermore, if properties (a)–(c) are satisfied, then starting from the initial state in which all the springs are relaxed, applying an increasing input (deformation)  $\varepsilon$ , and measuring the corresponding increasing output  $\sigma = \phi(\varepsilon)$ , one obtains the so called *loading curve*  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which completely defines the PI operator  $\mathcal{P}_\phi : C(0, T) \rightarrow C(0, T)$  on the class of all continuously varying inputs  $\varepsilon : [0, T] \rightarrow \mathbb{R}$  via an explicit formula involving  $\phi$  and a sequence of local running extremum values of  $\varepsilon$  (see (27) below). It is worth mentioning that the same measurement of the loading curve is used in nonlinear elasticity to identify the non-hysteretic relationship between  $\varepsilon$  and  $\sigma$  by the Nemytskii operator  $\sigma(t) = \phi(\varepsilon(t))$ .

The main premise of the of the phenomenological PI model is that Prandtl's elements do not interact. The model of Moreau addresses a much more general setting, in which an arbitrary spatial configuration of nodes connected by Prandtl's elastic-ideal plastic springs deforms quasistatically (the balance of forces at each node is zero at all times) in response to either (i) external forcing applied at a selected set of nodes or (ii) controlled variation of the distance between nodes for a selected set of pairs of nodes, or (iii) simultaneous application of the two types of inputs above. In this setting, the relationship between the vector of stresses of individual springs and the input is complicated and is described by the differential inclusion (or, equivalently, a variational inequality

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<sup>2</sup>Rate-independence means that the operator  $\mathcal{P}$  that maps the time series  $\varepsilon(t)$  of deformation to the time series  $\sigma(t)$  of stress commutes with any increasing transformation  $\tau(t)$  of time,  $\mathcal{P} \circ \tau = \tau \circ \mathcal{P}$ .

<sup>3</sup>This property is common for most phenomenological models of hysteresis with scalar-valued inputs and outputs including the Preisach model [6, 8, 9] and the Ising model. Manifestations of this property are also known as the *return point memory*, *wiping-out property*, *no passing rule*, and *Madelung's update rule*.

ity), which is known as the Moreau *sweeping process*. This process has a nice kinematic interpretation, in which variations of the input induce the motion and deformation of the convex domain  $Z = Z(t)$  of admissible stresses, and the set  $Z$  drags a point representing the vector of stresses according to a natural kinematic rule.

Here, we consider an (arbitrary) configuration of Prandtl's elastic-ideal plastic springs controlled by one scalar-valued input—the distance  $g(t)$  between two selected nodes,  $A$  and  $B$ . The corresponding sweeping process belongs to a special class known as a (multidimensional) play operator with a unidirectional input. Even for this setting, the relationship between the input  $g(t)$  and the stress  $\sigma_i(t)$  of any given spring can be quite complicated. In particular, hysteresis loops corresponding to a periodically varying  $g$  can be non-closed, thus violating property (b) of the PI operator stated above. On the other hand, there are multiple examples of topologies of the spring configuration for which the relationship between  $g$  and  $\sigma_i$  is a PI operator  $\mathcal{P}_{\phi_i}$  for every  $i$  (as well as the total reaction force at the node  $A$ , and at the node  $B$ , is related to  $g$  via a PI operator). In particular, such examples can be constructed using the composition property of PI operators.

In this paper, we consider the following question: Under which assumptions the Moreau sweeping process driven by a scalar-valued input  $g(t)$  is equivalent to a PI operator? The answer will be given in geometric terms involving a vector-valued analog of the loading curve, which we define for the sweeping process. We show that if this curve and the domain  $Z$  of admissible stresses satisfy simple geometric conditions, then the reaction force of the system of coupled Prandtl's elements responds to the input  $g(t)$  exactly in the same way as the reaction force of a system of decoupled Prandtl's elements of a certain effective PI model. Furthermore,  $\sigma_i(t) = (\mathcal{P}_{\phi_i}g)(t)$  for each spring of the Moreau model, where  $\phi_i$  is an appropriate projection of the loading curve of the Moreau model in the space of stresses. In other words, we perform a reduction of the more complex model of Moreau to a simpler affective PI model when such reduction is possible.

The paper is organized as follows. In the next section, we present an outline of models of Moreau and Prandtl-Ishlinskii. In Section 3, the main reduction theorem is proved. Section 4 contains some discussion. In particular, we show that any Moreau model obtained by a small perturbation of a PI model satisfies the conditions of the main theorem and hence is reducible to an effective PI model. The paper ends with a summary of results and a few concluding remarks.

## 2 The model of Moreau

### 2.1 Mechanical setting

Following the model of Moreau, let us consider an arrangement of  $N$  nodes, some of which are connected by elastic-ideal plastic springs, see Fig. 1. For simplicity,

we consider a one-dimensional arrangement (see Remark ??). That is, all the nodes lie on a straight line (the  $x$ -axis) and the springs, as well as spring forces, are elongated along this line at all times. The coordinate of node  $i$  will be denoted  $x_i$ . It is assumed that there is a configuration  $(x_1^*, \dots, x_N^*)$  called the zero configuration, in which the springs experience zero stress, and we consider deformations of the springs relative to this configuration. The deformation of the spring connecting nodes  $i$  and  $j$  is denoted by  $\varepsilon_{ij}$ :

$$\varepsilon_{ij} = x_j - x_i - (x_j^* - x_i^*), \quad (1)$$

and the force (stress) of this spring is denoted by  $\sigma_{ij}$ .

This arrangement of springs can be associated with a non-directed graph  $\Gamma$  with the nodes associated with graph's vertices and springs associated with graph's edges. Without loss of generality, this graph is assumed to be connected. By  $M$ , we denote the set of indices  $(ij)$  such that  $(ij) \in M$  whenever there is a spring connecting the nodes  $i$  and  $j$ . Here and henceforth, we agree that  $(ij) = (ji)$  and  $i \neq j$  for  $(ij) \in M$ .

The general Moreau sweeping process models a quasistatic response of the system of springs to external controls (inputs). There are two types of admissible controls: (a) an external force  $F_i = F_i(t)$  applied to a node  $i$ ; and, (b) a *moving affine constraint* that prescribes a distance  $L_{ij} = x_j^* - x_i^* + g_{ij}(t)$  between a pair of nodes  $i$  and  $j$  at all times. Multiple moving constraints and external forces may be applied simultaneously at several pairs of nodes and nodes, respectively. However, in this paper, we consider a system with one control. To be specific, we choose to consider a system of springs controlled by one moving constraint. Without loss of generality, we assume that this constraint defines the distance between nodes 1 and  $N$  and that these nodes are not connected by a spring:

$$\varepsilon_{1N} = g(t) \quad (2)$$

where  $g \in W^{1,1}$  is a given function of  $t \geq 0$  satisfying  $g(0) = 0$  and  $(1N) \notin M$ .

We note that Moreau's theory equally applies to configurations, where a pair of nodes can be connected by multiple springs. In this case,  $\Gamma$  is a multigraph.

## 2.2 Prandtl's elastic-ideal plastic spring

In the models of Moreau and Prandtl-Ishlinskii, the stress  $\sigma_{ij} = \sigma_{ij}(t)$  and deformation  $\varepsilon_{ij} = \varepsilon_{ij}(t)$  of each spring are related by Prandtl's nonlinear hysteretic constitutive law that combines an ideally elastic spring with a dry friction element, see Fig. 2. According to this constitutive law,

$$\sigma_{ij} = a_{ij}e_{ij}, \quad (3)$$

where  $a_{ij} > 0$  is the Young's modulus of the spring in the elastic domain, and the internal variable  $e_{ij}$  called *elastic deformation* is related to the deformation  $\varepsilon_{ij}$  by the so-called one-dimensional *stop* operator  $\mathcal{S}_{\rho_{ij}}$ :

$$e_{ij} = \mathcal{S}_{\rho_{ij}}[\varepsilon_{ij}]. \quad (4)$$

Here  $\rho_{ij}$  is the maximal value of the elastic deformation, which defines the maximal magnitude  $r_{ij} := a_{ij}\rho_{ij}$  of stress for the spring, *i.e.*

$$|\sigma_{ij}| \leq r_{ij} \quad (5)$$

at all times.

For piecewise monotone inputs  $\varepsilon_{ij}$ , relationship (4) between the time series of the deformation and the elastic deformation is given by the explicit formula

$$e_{ij}(t) = F_{\rho_{ij}}(\sigma_{ij}(t_n) + \varepsilon_{ij}(t) - \varepsilon_{ij}(t_n)), \quad t \in [t_n, t_{n+1}],$$

where  $0 = t_0 < t_1 < t_2 < \dots$  is a partition of the time domain into intervals of monotonicity of the input  $\varepsilon_{ij}$  and

$$F_{\rho}(x) = \begin{cases} -\rho, & x < -\rho, \\ x, & -\rho \leq x \leq \rho, \\ \rho, & x > \rho, \end{cases}$$

see Fig. 3. Furthermore, as shown in [6], operator (4) admits a continuous extension from the dense set of piecewise monotone inputs to the whole space in each of the spaces  $C$ ,  $W^{1,1}$ , and  $BV$ . In the space  $W^{1,1}$ , operator (4) can be alternatively defined by the variational inequalities

$$|e_{ij}(t)| \leq \rho_{ij}; \quad (\dot{e}_{ij}(t) - \dot{e}_{ij}(t))(e_{ij}(t) - z) \geq 0 \quad \text{for all } |z| \leq \rho_{ij}, \quad (6)$$

which should be satisfied for a.e.  $t$ . This law postulates the ideal elastic response of the spring as long as the stress remains in the range  $-r_{ij} < \sigma_{ij} < r_{ij}$ . If the maximal admissible value of the stress is reached but the deformation continues to increase in absolute value, the stress remains at its maximal value, *i.e.*  $\sigma_{ij} = r_{ij}$  for  $\dot{e}_{ij} > 0$  and  $\sigma_{ij} = -r_{ij}$  for  $\dot{e}_{ij} < 0$ , that is the spring *yields*.

Everywhere below, we assume that all the deformations and forces are initially zero, that is  $\varepsilon_{ij} = e_{ij} = 0$ ,  $\sigma_{ij} = 0$  for all  $(ij) \in M$  at the initial moment  $t_0 = 0$ .

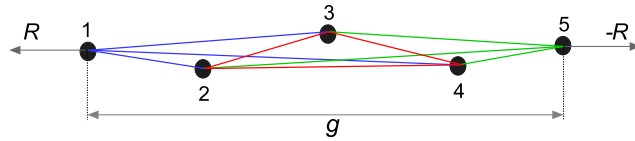


Figure 1: Connection of 5 nodes with 9 elastic-ideal plastic springs.

### 2.3 Configuration space

Define the vector  $\varepsilon$  of deformations  $\varepsilon_{ij}$  (where  $(ij) \in M$ ) of all the springs and the vector  $\sigma$  of their corresponding stresses  $\sigma_{ij}$ . Similarly,  $e$  denotes the vector of elastic deformations  $e_{ij}$ , and we will also use the vector

$$p := \varepsilon - e$$

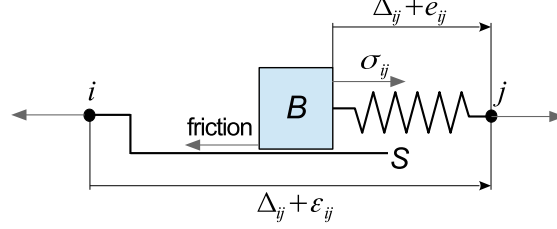


Figure 2: A nonlinear spring connecting two nodes in the model of Moreau can be represented as a combination of a dry friction element and an ideally elastic spring obeying the Hook's law (3). The deformation  $e_{ij}$  of the ideal spring is called elastic deformation;  $\varepsilon_{ij}$  is the elongation of the distance between the nodes  $i$  and  $j$  (with respect to the distance  $\Delta_{ij} = x_j^* - x_i^*$  at zero configuration). Dry friction between the box  $B$  and the surface  $S$  produces the friction force, which is opposite to the ideal spring force at all times (the quasistatic model). The magnitude of the friction force is limited by the maximal value  $a_{ij}\rho_{ij}$ . Therefore, the ideal spring deforms but the box does not move with respect to the surface  $S$  as long as  $|\sigma_{ij}| < a_{ij}\rho_{ij}$ . When  $|\sigma_{ij}| = a_{ij}\rho_{ij}$ , the deformation  $e_{ij}$  of the ideal spring and the force remain constant, while the box moves with respect to the surface in the direction of the spring force.

of *plastic deformations*  $p_{ij}$ . The model of Moreau defines dynamics in the configuration space  $\mathbb{E} = \mathbb{R}^m \ni \varepsilon, \sigma, e, p$ , where  $m = \#M$  is the total number of springs.

In what follows, the set of geometric constraints (1) with the additional affine moving constraint (2) will be expressed as the inclusion

$$\varepsilon(t) \in W + k_0 g(t), \quad t \geq 0, \quad (7)$$

with an appropriate choice of the subspace  $W$  of the configuration space  $\mathbb{E}$  and a vector  $k_0$ . We remark that geometric constraints in parametric form (1) can be equivalently expressed in terms of the cycles  $(i_1 i_2 \dots i_k)$  of the graph  $\Gamma$  with the added edge  $(1 N)$ . Namely, the sum of deformations along the edges  $(i_1 i_2), (i_2 i_3), \dots, (i_k i_1)$  of every cycle must be zero:

$$\varepsilon_{i_1 i_2} + \varepsilon_{i_2 i_3} + \dots + \varepsilon_{i_{k-1} i_k} + \varepsilon_{i_k i_1} = 0. \quad (8)$$

The main assumption of Moreau model is that deformations are quasistatic. Therefore, the balance of forces at each node is zero at any moment:

$$\sum_{(ij) \in M} \sigma_{ij} = 0 \quad \text{for every } i = 2, \dots, N-1, \quad (9)$$

where the summation is over  $j$ . The balance of forces at each of the nodes 1 and  $N$  connected by the moving constraint includes the (unknown) reaction  $R$  of this constraint:

$$R + \sum_{(1j) \in M} \sigma_{1j} = -R + \sum_{(Nj) \in M} \sigma_{Nj} = 0. \quad (10)$$

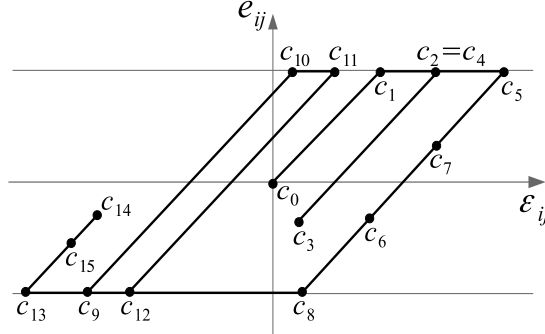


Figure 3: An example of an input-output trajectory  $c_0 c_1 \dots c_{15}$  of a nonlinear spring with  $c_n = (\varepsilon_{ij}(t_n), e_{ij}(t_n))$  and  $t_0 < t_1 < \dots$ . The time series of the deformation  $\varepsilon_{ij}$  and the elastic deformation  $e_{ij}$  are related by the stop operator  $e_{ij} = \mathcal{S}_{\rho_{ij}}[\varepsilon_{ij}]$ . Slanted lines have the slope 1; horizontal lines are  $e_{ij} = \pm \rho_{ij}$ .

Relations (10) follow from (9).

## 2.4 Moreau sweeping process

Let us briefly summarize the results of Moreau for the particular case of a system of springs with one control (2) that we are considering. For a closed convex set  $Z \subset \mathbb{E}$ , we denote by  $N_Z(z)$  the external normal cone to the set  $Z$  at a point  $z \in Z$ :

$$N_Z(z) = \{y \in \mathbb{E} : \langle y, z - \tilde{z} \rangle \geq 0 \text{ for all } \tilde{z} \in Z\}, \quad (11)$$

where the standard scalar product

$$\langle y, z \rangle = \sum_{(ij) \in M} y_{ij} z_{ij}, \quad y, z \in \mathbb{E}, \quad (12)$$

is used. Let us introduce the parallelepiped

$$C = \{\sigma \in \mathbb{E} : |\sigma_{ij}| \leq r_{ij}, (ij) \in M\}$$

of admissible stresses defined by (5). As was shown by J. J. Moreau, system (1)–(4), (9) is equivalent to the system

$$e + p \in W + k_0 g(t), \quad (13)$$

$$\sigma = Ae, \quad (14)$$

$$\sigma \in W^\perp \cap C, \quad (15)$$

$$\dot{p} \in N_C(\sigma) \quad (16)$$

with time dependent variables  $e, p, \sigma \in \mathbb{E} = \mathbb{R}^m$ . Here  $A$  is the diagonal positive matrix with the entries  $a_{ij}$  such that (14) is equivalent to (3); recall that  $e + p = \varepsilon$ , hence (13) is nothing else as the relationship (7), which is equivalent to the

set of geometric constraints (1) with the additional moving affine constraining (2); the inclusion  $\sigma \in W^\perp$ , where  $W^\perp$  denotes the orthogonal complement of  $W$  in the space  $\mathbb{E}$ , is equivalent to equations (9) that state the balance of forces; and, the differential inclusion  $\dot{p} \in N_C(\sigma)$ , where dot denotes differentiation with respect to time and  $\sigma \in C$ , is equivalent to the variational inequality (6) that expresses the constitutive law (4) of Prandtl's spring.

Following J. J. Moreau, it is convenient to use the rescaled variables

$$u = A^{\frac{1}{2}}e, \quad v = A^{\frac{1}{2}}p,$$

where the diagonal matrix  $A^{\frac{1}{2}}$  is the positive square root of  $A$ . Introducing the orthogonal subspaces

$$U := A^{\frac{1}{2}}W, \quad V := A^{-\frac{1}{2}}W^\perp \quad (17)$$

and the scaled parallelepiped

$$\Pi := A^{-\frac{1}{2}}C, \quad (18)$$

and noticing that

$$\langle \dot{p}, \sigma - \tilde{z} \rangle = \langle \dot{p}, Ae - \tilde{z} \rangle = \langle A^{\frac{1}{2}}\dot{p}, A^{\frac{1}{2}}e - A^{-\frac{1}{2}}\tilde{z} \rangle = \langle \dot{v}, u - A^{-\frac{1}{2}}\tilde{z} \rangle,$$

we see that equations (13)–(16) are equivalent to the system

$$\begin{aligned} u + v &\in U + A^{\frac{1}{2}}k_0g(t), \\ u &\in V \cap \Pi, \\ \dot{v} &\in N_\Pi(u). \end{aligned}$$

Now, introducing the orthogonal projection

$$f_0 = \text{proj}_V A^{\frac{1}{2}}k_0 \quad (19)$$

of the vector  $A^{\frac{1}{2}}k_0$  on the subspace  $V = U^\perp$ , and using the new variables

$$s = u - f_0g(t), \quad \xi = u + v - f_0g(t),$$

one can rewrite this system as

$$\xi \in U, \quad (20)$$

$$s \in V \cap \Pi - f_0g(t), \quad (21)$$

$$\dot{\xi} - \dot{s} \in N_{\Pi - f_0g(t)}(s). \quad (22)$$

Finally, combining inclusion (22) with  $-\dot{\xi} \in U = V^\perp = N_V(s)$  and using the identity  $N_{D_1}(s) + N_{D_2}(s) = N_{D_1 \cap D_2}(s)$ , we arrive at the differential inclusion

$$-\dot{s} \in N_{\Pi \cap V - f_0g(t)}(s), \quad (23)$$

which is known as the *Moreau sweeping process* with the characteristic set (input)  $Z(t) = \Pi \cap V - f_0g(t)$ .



A few remarks are in order. First,  $\Pi \cap V$  is a centrally symmetric convex polytope. Its central symmetry is important for the following discussion. Second, the characteristic set  $Z(t) = \Pi \cap V - f_0 g(t)$  of the Moreau process is obtained as time-dependent shift of this polytope. The Moreau process with a characteristic set of the form  $Z(t) = Z_0 + y(t)$ , where  $y$  is a single-valued function  $y : \mathbb{R}_+ \rightarrow V$  and  $Z_0 \subset V$  is a convex set, is known as the multi-dimensional play operator with input  $y$  [6]. In our case, the input  $y(t) = -f_0 g(t)$  has a fixed direction  $f_0 \in V$ , *i.e.* we consider the Moreau process of the play type, which in effect has a one-dimensional input. In general, for a system with multiple inputs (moving constraints and external forces), the shape of the characteristic set  $Z(t)$  may change with time, *i.e.*  $Z : \mathbb{R}_+ \rightarrow 2^V$  is a more general regular set-valued convex-valued function.

Eq. (23) is equivalent to the differential inclusion

$$-\dot{u} + f_0 \dot{g}(t) \in N_{\Pi \cap V}(u) \quad (24)$$

known as the multi-dimensional stop operator with the input  $f_0 g(t)$ . This differential inclusion coupled with the initial condition  $u(0) = 0$  has a unique solution  $u \in W^{1,1}(0, T; V)$  for any input  $g \in W^{1,1}(0, T; \mathbb{R})$ ; regularity properties of the solution operator that maps  $g$  to  $u$  in spaces  $W^{1,1}$ ,  $C$ , and  $BV$  are well understood. Furthermore, system (20) – (22) coupled with the initial conditions  $s(0) = \xi(0) = 0$  (we also assume  $g(0) = 0$ ) has a solution  $(\xi, s)$ . While forces of the springs (the component  $s$ ) are defined uniquely for a given input  $g = g(t)$ , simple examples show that the solution  $(\xi, s)$  may be non-unique, that is deformations are not necessarily uniquely defined<sup>4</sup>. It should be noted that examples with multiple solutions are non-generic. However, the author is not aware of results that would establish uniqueness of deformations under genericity assumptions.

## 2.5 Example

Consider the system of springs shown in Fig. 1. The configuration space  $\mathbb{E} = \mathbb{R}^9$  of this system consists of vectors  $\varepsilon = (\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \varepsilon_{23}, \varepsilon_{24}, \varepsilon_{25}, \varepsilon_{34}, \varepsilon_{35}, \varepsilon_{45})'$  (where prime denotes the transposition), which satisfy the following set of geometric constraints:

$$\varepsilon_{23} = \varepsilon_{13} - \varepsilon_{12}, \quad \varepsilon_{24} = \varepsilon_{14} - \varepsilon_{12}, \quad \varepsilon_{25} = \varepsilon_{15} - \varepsilon_{12}, \quad \varepsilon_{34} = \varepsilon_{14} - \varepsilon_{13}, \quad \varepsilon_{35} = \varepsilon_{15} - \varepsilon_{13},$$

and  $\varepsilon_{45} = \varepsilon_{15} - \varepsilon_{14}$  (cf. (8)). We assume that the system is subject to the additional moving affine constraint  $\varepsilon_{15} = g(t)$ . Combining all the constraints, we obtain

$$\varepsilon = \varepsilon_{12} k_1 + \varepsilon_{13} k_2 + \varepsilon_{14} k_3 + g(t) k_0,$$

where

$$\begin{aligned} k_1 &= (1, 0, 0, -1, -1, -1, 0, 0, 0)'; & k_2 &= (0, 1, 0, 1, 0, 0, -1, -1, 0)'; \\ k_3 &= (0, 0, 1, 0, 1, 0, 1, 0, -1)'; & k_0 &= (0, 0, 0, 0, 0, 1, 0, 1, 1)'. \end{aligned}$$

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<sup>4</sup>The simplest example is a system of two identical springs connecting nodes 1 and 2 and nodes 2 and 3, respectively, with the moving constraint applied to nodes 1 and 3.

This equation is equivalent to relation (7) with  $W = \text{span}(k_1, k_2, k_3)$ . The balance of forces (9) at the nodes 2, 3, 4 reads

$$\langle \sigma, k_1 \rangle = \langle \sigma, k_2 \rangle = \langle \sigma, k_3 \rangle = 0$$

or, equivalently,  $\sigma(t) \in W^\perp$  with  $\sigma = (\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{34}, \sigma_{35}, \sigma_{45})'$ . The matrix  $A$  has the form

$$A = \text{diag}(\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{34}, \alpha_{35}, \alpha_{45}).$$

In this example, the set  $\Pi \cap V$  is a 6-dimensional centrally symmetric convex polytope.

## 2.6 The Prandtl-Ishlinskii model

The Prandtl-Ishlinskii operator  $\mathcal{P}$  has scalar-valued inputs and outputs and is obtained as a weighted sum of a finite number of one-dimensional stop operators:

$$R = \mathcal{P}[g] := \sum_{n=1}^K \bar{a}_n \mathcal{S}_{\bar{\rho}_n}[g], \quad (25)$$

where  $g = g(t)$  and  $R = R(t)$  are continuous input and output, respectively;  $\bar{a}_n$  is the Young's modulus of the  $n$ -th spring in the elastic domain;  $\bar{r}_n := \bar{a}_n \bar{\rho}_n$  is the maximal stress of the spring<sup>5</sup>; and, without loss of generality we can assume that

$$0 < \bar{\rho}_1 < \bar{\rho}_2 < \dots < \bar{\rho}_K.$$

This operator is associated with the connection of Prandtl's springs shown in Fig. 4(a), i.e.  $\Gamma$  consists of two vertices (nodes) 1 and 2 connected by a multiedge. Formula (25) relates the elongation  $g = x_2 - x_2^* - (x_2^* - x_1^*)$  of the distance between the nodes (relative to the zero configuration) with the reaction force  $R$  applied at node 2; the reaction at node 1 equals  $-R$ .

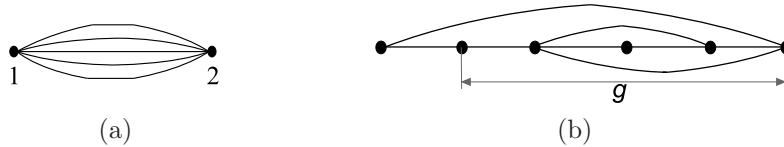


Figure 4: (a) Parallel connection of springs in the Prandtl-Ishlinskii model. (b) A reducible connection of springs.

It is easy to see that the same relationship (25) between the elongation  $g$  of the moving constraint and the reaction force  $R$  is valid for any simple connected

<sup>5</sup>The Prandtl-Ishlinskii model can include infinitely many stops, in which case the sum in (25) is replaced by the integral  $R = \int_0^\infty a(\rho) \mathcal{S}_\rho[g] d\rho$ . However, in this work, only the finite (discretized) model (25) is considered.

graph  $\Gamma$  of order  $N$  in which every vertex  $i = 2, \dots, N-1$  has degree 2 and the moving affine constraint is applied to the distance between the nodes 1 and  $N$  as in (2). Such connections will be called *linear*, see Fig. 5. For linear connections, the parameters  $K, \bar{a}_n, \bar{\rho}_n$  in (25) are “effective” quantities that can be related to the parameters  $a_{ij}, \rho_{ij}$  of the springs (Young’s moduli and yielding thresholds) via simple formulas. Indeed, one can show that the stress  $\sigma_{ij} = a_{ij} \mathcal{S}_{\rho_{ij}}[\varepsilon_{ij}]$  of each spring is related to the controlled distance  $g = g(t)$  between the nodes 1 and  $N$  by another stop operator  $\sigma_{ij} = \tilde{a}_{ij} \mathcal{S}_{\tilde{\rho}_{ij}}[g]$  with appropriate effective  $\tilde{a}_{ij}$  and  $\tilde{\rho}_{ij}$ . For example, for the connection shown in Fig. 5,

$$\tilde{a}_{12} = \tilde{a}_{23} = \tilde{a}_{38} = \frac{1}{\frac{1}{a_{12}} + \frac{1}{a_{23}} + \frac{1}{a_{38}}}, \quad \tilde{a}_{14} = \tilde{a}_{48} = \frac{1}{\frac{1}{a_{14}} + \frac{1}{a_{48}}},$$

$$\tilde{a}_{15} = \tilde{a}_{56} = \tilde{a}_{67} = \tilde{a}_{78} = \frac{1}{\frac{1}{a_{15}} + \frac{1}{a_{56}} + \frac{1}{a_{67}} + \frac{1}{a_{78}}}$$

and

$$\tilde{r}_{12} = \tilde{r}_{23} = \tilde{r}_{38} = \min\{r_{12}, r_{23}, r_{38}\}, \quad \tilde{r}_{14} = \tilde{r}_{48} = \min\{r_{14}, r_{48}\},$$

$$\tilde{r}_{15} = \tilde{r}_{56} = \tilde{r}_{67} = \tilde{r}_{78} = \min\{r_{15}, r_{56}, r_{67}, r_{78}\},$$

where  $r_{ij} = a_{ij} \rho_{ij}$ ,  $\tilde{r}_{ij} = \tilde{a}_{ij} \tilde{\rho}_{ij}$ . Equations (10) now imply that  $g$  is mapped to the reaction force  $R$  of the constraint by the Prandtl-Ishlinskii operator  $R(t) = \mathcal{P}[g](t) = \tilde{a}_{38} \mathcal{S}_{\tilde{\rho}_{38}}[g] + \tilde{a}_{48} \mathcal{S}_{\tilde{\rho}_{48}}[g] + \tilde{a}_{78} \mathcal{S}_{\tilde{\rho}_{78}}[g]$ . The general linear connection is considered in Section 4.

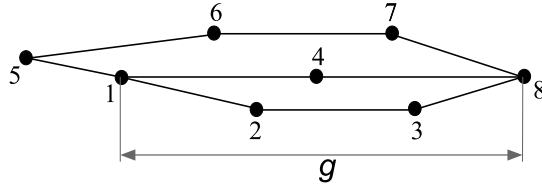


Figure 5: A “linear” connection of springs, which is equivalent to the Prandtl-Ishlinskii model.

An important characterization of Prandtl-Ishlinskii operator (25) is the so-called *loading curve*  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined as the response to the linear input:

$$\phi = \mathcal{P}[g_{id}] \quad \text{with} \quad g_{id}(t) := t, \quad t \geq 0$$

(where, in accordance with our agreement, we assume that  $\sigma_n(0) = 0$  for all the stop operators  $\sigma_n = \mathcal{S}_{\tilde{\rho}_n}[g_{id}]$  in (25)). Clearly,  $\phi$  is a piecewise linear function,

which satisfies  $\phi(0) = 0$  and  $\phi(\alpha) = \text{const}$  for sufficiently large  $\alpha$ . Equivalently,

$$\phi(0) = 0, \quad \frac{d\phi}{d\tau}(\tau) = \begin{cases} \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_K, & 0 \leq \tau < \bar{\rho}_1, \\ \bar{a}_2 + \cdots + \bar{a}_K, & \bar{\rho}_1 \leq \tau < \bar{\rho}_2, \\ \vdots & \\ \bar{a}_K, & \bar{\rho}_{K-1} \leq \tau < \bar{\rho}_K, \\ 0, & \tau > \bar{\rho}_K. \end{cases} \quad (26)$$

The loading curve uniquely identifies operator (25) [6], therefore we will write  $\mathcal{P} = \mathcal{P}_\phi$  when needed. Furthermore, for any  $g = g(t) \in C$  with  $g(0) = 0$ , the output  $R \in C$  defined by (25) can be expressed explicitly in terms of the function  $\phi$  as follows:

$$R(t) = \phi(G_1(t)) + 2 \sum_{k \geq 2} \phi \left( \frac{G_k(t) - G_{k-1}(t)}{2} \right), \quad t > 0, \quad (27)$$

where the so-called *running main extremum values*  $G_k$  of  $g$  are defined by the relationships

$$G_0(t) := \max\{|g(\tau)| : \tau \in [0, t]\}; \\ \tau_1 := \max\{\tau \in [0, t] : |g(\tau)| = G_0\}; \quad G_1 := g(\tau_1)$$

and, if  $G_1 \leq 0$ , then

$$G_{2i}(t) := \max\{g(\tau) : \tau \in [\tau_{2i-1}, t]\}; \quad \tau_{2i} := \max\{\tau \in [0, t] : g(\tau) = G_{2i}\}; \\ G_{2i+1}(t) := \min\{g(\tau) : t \in [\tau_{2i}, t]\}; \quad \tau_{2i+1} := \max\{\tau \in [0, t] : g(\tau) = G_{2i+1}\};$$

if  $G_1 > 0$ , then

$$G_{2i}(t) := \min\{g(\tau) : \tau \in [\tau_{2i-1}, t]\}; \quad \tau_{2i} := \max\{\tau \in [0, t] : g(\tau) = G_{2i}\}; \\ G_{2i+1}(t) := \max\{g(\tau) : t \in [\tau_{2i}, t]\}; \quad \tau_{2i+1} := \max\{\tau \in [0, t] : g(\tau) = G_{2i+1}\}$$

for  $i = 1, 2, \dots$

By definition, a linear combination of operators (25) with non-negative coefficients is also a Preisach-Ishlinskii operator,  $c_1 \mathcal{P}_{\phi_1} + c_2 \mathcal{P}_{\phi_2} = \mathcal{P}_{c_1 \phi_1 + c_2 \phi_2}$ . Remarkably, the class of the Prandtl-Ishlinskii operators (25) is also closed with respect to composition due to the identity  $\mathcal{P}_{\phi_1} \circ \mathcal{P}_{\phi_2} = \mathcal{P}_{\phi_1 \circ \phi_2}$  [1, 7]<sup>6</sup>. Using these properties, one can associate a Prandtl-Ishlinskii operator with more complex topologies  $\Gamma$  than a linear connection such as in Fig. 5. In particular, for a multigraph with vertices  $1, \dots, N$ , let us define an elementary graph operation that replaces a multiedge with a simple edge; an elementary operation that replaces a vertex  $i \notin \{1, N\}$  of degree 2 and the two edges emanating from it with one edge; and, an elementary operation that eliminates a vertex  $i \notin \{1, N\}$  of degree 1 and the edge emanating from it from the graph. We call a connected graph  $G$  of order  $N$  *reducible* if a finite number of such operations

<sup>6</sup>The composition formula for Prandtl-Ishlinskii operators is based on Brokate's formula  $(\text{Id} - \mathcal{S}_{\rho_1}) \circ (\text{Id} - \mathcal{S}_{\rho_2}) = (\text{Id} - \mathcal{S}_{\rho_1 + \rho_2})$  for the stop and play operators [7].

can reduce it to the trivial graph that consists of two vertices 1 and  $N$  and an edge between them<sup>7</sup>, see Fig. 4(b). One can show that if a connection of springs with a reducible graph  $G$  is driven by one moving constraint (2), then the elongation  $g$  of the constraint is mapped to the stress of every spring by an (effective) Prandtl-Ishlinskii operator,  $\sigma_{ij} = \mathcal{P}_{\phi_{ij}}[g]$  and the reaction force is given by (25) [3].

### 3 Main result

Consider a graph  $\Gamma$  with edges  $(ij) \in M$  and the corresponding system of  $m = \#M$  connected Prandtl's springs with parameters  $a_{ij}$  (stiffness) and  $r_{ij}$  (maximal stress), see Sections 2.1, 2.2. Assume that the system is driven by one moving affine constraint (2) and the set of all the constraints is described by inclusion (7).

Denote by  $u^*(t) : [0, L] \rightarrow \Pi \cap V$  the solution of the differential inclusion (24) with the input  $g(t) = t$  and the initial value  $u^*(0) = 0$ . The trajectory  $\gamma$  of this solution is a polyline  $B_0 B_1 \cdots B_\ell$  with one end  $B_0$  at the origin. Each link of this polyline belongs to a different face of the polytope  $\Pi \cap V$ , that is  $B_{k-1} B_k \subset F_{k-1}$  for  $k = 1, \dots, \ell$ , where  $F_k$  are (closed) faces of the polytope, which are all different, and  $F_0 = \Pi \cap V$ . Denote by  $E_k$  the minimal affine subspace of  $V$  which contains the face  $F_k$  and by  $\overset{\circ}{F}_k$  the interior of the set  $F_k$  in  $E_k$ .

The parametrization  $u^*(t)$  of the polyline  $\gamma$  defines the partition  $0 = d_0 < d_1 < \dots < d_\ell = L$  of the interval  $[0, L]$  by preimages of the points  $B_k$ :

$$u^*(0) = B_0 = 0, \quad u^*(d_1) = B_1, \quad u^*(d_2) = B_2, \quad \dots, \quad u^*(d_\ell) = B_\ell. \quad (28)$$

**Theorem 1.** *Suppose that*

$$F_{\ell-1} \subset F_{\ell-2} \subset \dots \subset F_0; \quad B_k \in \overset{\circ}{F}_k \quad \text{for } k = 0, 1, \dots, \ell - 1; \quad (29)$$

*and*

$$\dim F_0 - \dim F_1 = \dim F_1 - \dim F_2 = \dots = \dim F_{\ell-2} - \dim F_{\ell-1} = 1. \quad (30)$$

*Suppose that the parallelepiped*

$$\Omega = \{y = \tau_1 B_0 B_1 + \dots + \tau_\ell B_{\ell-1} B_\ell, \quad |\tau_i| \leq 1, \quad i = 1, \dots, \ell\} \quad (31)$$

*belongs to the polytope  $\Pi \cap V$ . Assume that the map  $u^* : [0, L] \rightarrow \gamma$  is invertible. Then, for every input  $g(t)$  satisfying  $g(0) = 0$  and  $|g(t)| \leq L$  for all  $t \geq 0$ , the stress of each spring relates to the variable  $g$  via the Preisach operator*

$$\sigma_{ij}(t) = \mathcal{P}_{\phi_{ij}}[g](t), \quad (ij) \in M, \quad (32)$$

---

<sup>7</sup>Using analogy with electrical circuits of resistors, a reducible graph corresponds to a circuit that can be solved by applying parallel and series connection rules.

with the loading curve

$$\phi_{ij}(\tau) = \sqrt{a_{ij}} u_{ij}^*(\tau), \quad 0 \leq \tau \leq L, \quad (33)$$

where  $u_{ij}^*$  is the  $(ij)$ -th component of the vector-valued function  $u^* : [0, L] \rightarrow \Pi$ .

As we establish below, under the conditions of this theorem, the Moreau sweeping process (24) behaves as an analog of the Prandtl-Ishlinskii operator with vector-valued outputs and the vector-valued  $u^* : [0, L] \rightarrow \Pi \cap V$  acts as a counterpart of the loading curve for this operator. More precisely, the solution of (24) with the zero initial condition  $u(0) = 0$  is given by the counterpart of formula (27):

$$u(t) = u^*(G_1(t)) + 2 \sum_{i \geq 2} u^* \left( \frac{G_i(t) - G_{i-1}(t)}{2} \right), \quad t > 0, \quad (34)$$

where  $G_k$  are running main extremum values of the scalar-valued input  $g$ ; and, we extend the function  $u^*$  to the interval  $[-L, L]$  by setting

$$u^*(-d) = -u^*(d), \quad d \in [0, L]. \quad (35)$$

Formulas (32), (33) immediately follow from (34) and the relationship  $\sigma = A^{\frac{1}{2}}u$ .

*Proof of Theorem 1.* Given an input  $g \in W^{1,1}([0, T]; \mathbb{R})$  with  $\max |g| \leq L$ , we need to show that the function defined by (34) satisfies  $u(t) \in \Pi \cap V$  for all  $t \in [0, T]$  and the inclusion (24) for almost every  $t \in [0, T]$ . Suppose for definiteness that  $G_1(t) \geq 0$ . Since

$$u^*(d) = \begin{cases} B_0 B_1 / (d_1 - d_0), & d_0 < d < d_1, \\ B_1 B_2 / (d_2 - d_1), & d_1 < d < d_2, \\ \vdots \\ B_{\ell-1} B_\ell / (d_\ell - d_{\ell-1}), & d_{\ell-1} < d < d_\ell, \end{cases} \quad (36)$$

with  $d_k$  defined in (28) (recall that  $d_0 = 0$ ), formula (34) with  $G_1(t) \geq 0$  implies the relation

$$u(t) = \sum_{k=1}^{\ell} \xi_k(t) \frac{B_{k-1} B_k}{d_k - d_{k-1}}, \quad t \geq 0, \quad (37)$$

with

$$\xi_k(t) = (\min\{G_1, d_k\} - d_{k-1})^+ + \sum_{i \geq 2} (-1)^{i-1} (\min\{|G_i - G_{i-1}|, 2d_k\} - 2d_{k-1})^+, \quad (38)$$

where  $a^+ = \min\{a, 0\}$  and  $G_i = G_i(t)$ . From the definition of the sequence  $G_i$  given below Eq. (27), it follows that

$$2G_1 \geq |G_2 - G_1| \geq |G_3 - G_2| \geq \dots \geq |G_i - G_{i-1}| \geq \dots \quad (39)$$

where only a finite number of differences  $G_i - G_{i-1}$  are nonzero. Therefore, (38) implies

$$|\xi_k| \leq d_k - d_{k-1}, \quad k = 1, \dots, \ell - 1,$$

and from (37) it follows that  $u(t)$  belongs to the parallelepiped  $\Omega$ , which by assumption belongs to  $\Pi \cap V$ . This proves  $u(t) \in \Pi \cap V$  for the function (34) in the case  $G_1(t) \geq 0$ . As the function  $u^*$  in (34) is odd, it is easy to see that the same inclusion is valid for  $G_1(t) < 0$ .

It remains to prove (24). As the solution operator of the Moreau sweeping process (23) subject to the zero initial condition  $u(0) = 0$  is continuous in the space  $W^{1,1}([0, T]; V)$  and so is the Prandtl-Ishlinskii operator in the space  $W^{1,1}([0, T]; \mathbb{R})$ , it suffices to show that function (34) satisfies (24) for piecewise linear continuous inputs  $g \in W^{1,1}([0, T]; \mathbb{R})$ . Furthermore, given such an input, it suffices to establish that if the inclusion (24) is valid for a.e.  $t$  from an interval  $[0, \theta] \subset [0, T]$ , then there is a  $\delta > 0$  such that this inclusion is also true almost everywhere in  $[\theta, \theta + \delta]$ .

In order to establish (34) is a solution of (24) on an initial small interval  $[0, \theta]$ , we recall the *rate-independence* property of the Moreau process and the Prandtl-Ishlinskii model, which means that the input-output operator of each model commutes with increasing transformations of time [?]. Let us take a sufficiently small  $\theta > 0$  so that  $g$  is linear on  $[0, \theta]$  and the trajectory  $u^*(g(t))$ , which satisfies  $u^*(g(0)) = u^*(0) = 0$ , remains in the interior of the polytope  $\Pi \cap V$  for all  $t \in [0, \theta]$ . Then, (34) is equivalent to  $u(t) = u^*(g(t))$ , which agrees with the rate-independence property of the Moreau process for an increasing  $g$ . Since the function  $u^*$  and the solution operator of the Moreau process are odd, (34) defines a solution of (24) on a sufficiently small interval  $[0, \theta]$  for a decreasing  $g$  too.

Assuming that (34) satisfies (24) for  $t \in [0, \theta]$ , let  $\hat{G}_k = G_k(\theta)$  be the sequence of the main extremum values of  $g$  at the moment  $\theta$ . Due to the central symmetry of  $\Pi$ , we can assume without loss generality that  $\hat{G}_1 > 0$ . Since  $g$  is piecewise linear, from the definition of  $\hat{G}_k$  it follows that there is an integer  $k_0$  such that  $\hat{G}_k = g(\theta)$  for all  $k \geq k_0$  and, depending on the parity of  $k_0$ , either

$$\hat{G}_2 < \hat{G}_4 < \dots < \hat{G}_{2p_0-2} < g(\theta) = \hat{G}_{2p_0-1} < \hat{G}_{2p_0-3} < \dots < \hat{G}_3 < \hat{G}_1 \quad (40)$$

with  $k_0 = 2p_0 - 1$ , or

$$\hat{G}_2 < \hat{G}_4 < \dots < \hat{G}_{2p_0} = g(\theta) < \hat{G}_{2p_0-1} < \hat{G}_{2p_0-3} < \dots < \hat{G}_3 < \hat{G}_1 \quad (41)$$

with  $k_0 = 2p_0$ . The argument for both cases is similar, and we assume for definiteness that (40) holds.

Since  $g$  is piecewise linear,  $\dot{g} = \text{const}$  on some interval  $(\theta, \theta + \delta)$ . Using the rate-independence of the Moreau process, it suffices to consider the cases  $\dot{g} = 1$ ,  $\dot{g} = -1$ , and  $\dot{g} = 0$  for  $t \in (\theta, \theta + \delta)$ . The latter case is trivial as  $g = \text{const}$  implies that  $u = \text{const}$  and  $G_k = \text{const}$  in (34) for all  $k$  and all  $t \in [\theta, \theta + \tau]$ . The cases when  $g$  increases ( $\dot{g} = 1$ ) and  $g$  decreases ( $\dot{g} = -1$ ) on  $(\theta, \theta + \tau)$  will be considered separately.

Let us introduce some notation. Suppose that the polytope  $\Pi \cap V$  is the intersection of subspaces

$$\bigcap_{i=1}^m \{x \in V : \langle n_i, x \rangle \leq c_i\} = \Pi \cap V,$$

where each hyperplane  $\langle n_i, x \rangle = c_i > 0$  contains one facet of  $\Pi \cap V$  and  $n_i$  is the unit outward normal vector to this facet. Condition (30) ensures that the face  $F_k$  of  $\Pi \cap V$  belongs to the intersection of  $k$  hyperplanes  $\langle n_i, x \rangle = c_i$ , hence we can number these hyperplanes in such a way that

$$F_k \subset E_k = \bigcap_{i=1}^k \{x \in V : \langle n_i, x \rangle = c_i\}, \quad k = 1, \dots, \ell - 1.$$

Recall that  $\overset{\circ}{F}_k$  the interior of the face  $F_k$  in the affine subspace  $E_k$  (with the agreement that  $E_0 = V$ ). With this notation, the outward normal cone to  $\Pi \cap V$  on  $\overset{\circ}{F}_k$  coincides with the positive linear span of the vectors  $n_1, \dots, n_k$ :

$$N_{\Pi \cap V}(x) = \left\{ \sum_{i=1}^k \alpha_i n_i : \alpha_1, \dots, \alpha_k \geq 0 \right\}, \quad x \in \overset{\circ}{F}_k, \quad k = 1, \dots, \ell - 1.$$

In particular, since relations (29) ensure that  $u^*(d) \in \overset{\circ}{F}_{k-1}$  for  $d_{k-1} < d < d_k$ , the inclusion (24) for  $\dot{u}^*$  is equivalent to

$$-\dot{u}^*(d) + f_0 \in \left\{ \sum_{i=1}^{k-1} \alpha_i n_i : \alpha_1, \dots, \alpha_{k-1} \geq 0 \right\} \quad \text{for } d_{k-1} < d < d_k$$

with  $k = 1, \dots, \ell - 1$ . Combining these relations with (36), we obtain

$$-\frac{B_{k-1}B_k}{d_k - d_{k-1}} + f_0 \in \left\{ \sum_{i=1}^{k-1} \alpha_i n_i : \alpha_1, \dots, \alpha_{k-1} \geq 0 \right\}. \quad (42)$$

One can also see that the inclusion (24) for  $u \in \overset{\circ}{F}_k$  is equivalent to the equality

$$\dot{u} = \dot{g}f_k, \quad (43)$$

where

$$f_k = \text{proj}_{\overset{\circ}{E}_k} f_0$$

is the orthogonal projection of the vector  $f_0$  onto the subspace  $\overset{\circ}{E}_k = \{x \in V : \langle n_i, x \rangle = 0, i = 1, \dots, k\}$ , which is parallel to  $E_k$  in  $V$ . Note that since  $U = V^\perp$ ,

$$f_k = \text{proj}_{\overset{\circ}{E}_k} A^{\frac{1}{2}} k_0.$$



Now, suppose that (40) holds at the moment  $\theta$  and  $\dot{g} = -1$  for  $t \in (\theta, \theta + \delta)$ . If  $\delta > 0$  is sufficiently small, then according to the definition of the main extremum values of  $g$ , one has

$$G_2(t) < \cdots < G_{2p_0-2}(t) < G_{2p_0}(t) = g(t) < G_{2p_0-1}(t) < G_{2p_0-3}(t) < \cdots < G_1(t)$$

with

$$G_1(t) = G_1(\theta), G_2(t) = G_2(\theta), \dots, G_{2p_0-1}(t) = G_{2p_0-1}(\theta) \text{ for all } t \in [\theta, \theta + \delta]$$

(cf. (41)). Therefore, comparing formula (34) at the moments  $\theta$  and  $t \in (\theta, \theta + \delta)$  (note that the sum in (34) contains nonzero terms for  $2 \leq k \leq 2p_0 - 1$  at the moment  $\theta$  and for  $2 \leq k \leq 2p_0$  at the moment  $t$ ), we obtain

$$u(t) = u(\theta) + 2u^* \left( -\frac{(t - \theta)f_0}{2} \right),$$

and using (35), (36) and  $\delta < d_1$ , we arrive at the relation

$$\dot{u} = -f_0, \quad t \in [\theta, \theta + \delta],$$

which agrees with (24).

Finally, suppose that  $\dot{g} = 1$  for  $t \in (\theta, \theta + \delta)$  and, again, relations (40) are valid at the moment  $\theta$ . In this case, assuming that  $\delta > 0$  is sufficiently small,

$$G_2(t) < \cdots < G_{2p_0-2}(t) < g(t) = G_{2p_0-1}(t) < G_{2p_0-3}(t) < \cdots < G_1(t)$$

with

$$G_1(t) = G_1(\theta), G_2(t) = G_2(\theta), \dots, G_{2p_0-2}(t) = G_{2p_0-2}(\theta) \text{ for all } t \in [\theta, \theta + \delta].$$

Therefore, (34) implies

$$u(t) = u^*(\theta) - 2u^* \left( \frac{G_{2p_0-1}(\theta) - G_{2p_0-2}(\theta)}{2} \right) + 2u^* \left( \frac{g(t) - G_{2p_0-2}(\theta)}{2} \right),$$

hence

$$\dot{u}(t) = \dot{u}^* \left( \frac{g(t) - G_{2p_0-2}(\theta)}{2} \right), \quad t \in (\theta, \theta + \delta). \quad (44)$$

Now, note that there is a  $k$ ,  $1 \leq k \leq \ell$ , such that

$$2d_{k-1} \leq G_{2p_0-1}(\theta) - G_{2p_0-2}(\theta) < 2d_k,$$

and since  $g(t) > G_{2p_0-2}(\theta)$ , one also has

$$2d_{k-1} < g(t) - G_{2p_0-2}(\theta) < 2d_k, \quad t \in (\theta, \theta + \delta), \quad (45)$$

for a sufficiently small  $\delta$ . Combining these relations with (36) and (44), we see that

$$\dot{u}(t) = \frac{B_{k-1}B_k}{d_k - d_{k-1}}, \quad t \in (\theta, \theta + \delta). \quad (46)$$

Also, relations (39) and (45) imply

$$2G_1 \geq |G_2 - G_1| \geq \dots \geq |g(t) - G_{2p_0-2}| > 2d_{k-1} \geq 2d_j, \quad j = 1, \dots, k-1,$$

where  $G_i = G_i(\theta) = G_i(t)$  for  $i \leq 2p_0 - 2$  and  $g(t) = G_{2p_0-1}(t)$ . Hence, for  $j = 1, \dots, k-1$ , (38) becomes

$$\xi_j(t) = d_j - d_{j-1} + 2 \sum_{i=2}^{2p_0-1} (-1)^{i-1} (d_j - d_{j-1}) = d_j - d_{j-1}, \quad t \in (\theta, \theta + \delta),$$

(note that all terms with  $i \geq 2p_0$  in the sum in (38) are zero) and (37) implies

$$u(t) = B_{k-1} + \sum_{j=k}^{\ell} \xi_k(t) \frac{B_{k-1} B_k}{d_k - d_{k-1}}, \quad t \in (\theta, \theta + \delta).$$

Since  $B_{k-1}, \dots, B_{\ell} \in F_{k-1}$ , it follows that  $u(t) \in F_{k-1}$  and hence

$$n_1, \dots, n_{k-1} \in N_{\Pi \cap V}(u(t)), \quad t \in (\theta, \theta + \delta).$$

Combining this relation with (42) and (46), we conclude that (24) with  $\dot{g} = 1$  holds for  $t \in (\theta, \theta + \delta)$ . This completes the proof.

## 4 Discussion

A few remarks are in order.

**1.** Under the assumptions Theorem 1, the complex connection of springs defined by the graph  $\Gamma$  responds to arbitrary variations  $g(t)$  of the given moving affine constraint in the same way as a simple parallel connection of springs described by an effective Prandtl-Ishlinskii operator. Indeed, the stress of each spring of the complex system is related to  $g(t)$  via the Prandtl-Ishlinskii operator (32) and the total force applied to the system at node 1 relates to the displacement  $g$  via the Prandtl-Ishlinskii operator (cf. (10))

$$-R = \sum_{(1j) \in M} \sigma_{1j} = \sum_{(1j) \in M} \mathcal{P}_{\phi_{1j}}[g](t).$$

**2.** According to formula (34), under the conditions of Theorem 1, hysteresis loops of the Moreau model and the loops of the Prandtl-Ishlinskii model have similar properties and can be constructed by the same simple manipulations with the graph of the loading curve, see [3] for details. The only difference is that this graph is two-dimensional for the Prandtl-Ishlinskii model, while the loading curve  $\gamma$  for the Moreau model in Theorem 1 is multi-dimensional.

**3.** Fig. 6 illustrates the role of conditions  $\Omega \subset \Pi \cap V$  and  $\dim F_{k-1} - \dim F_k = 1$  of Theorem 1. Fig. 6(a) violates condition (31) because  $\Omega \not\subset \Pi \cap V$ . In Fig. 6(b),  $\dim F_1 = \dim F_2$ , hence conditions (29)–(30) are violated. In both

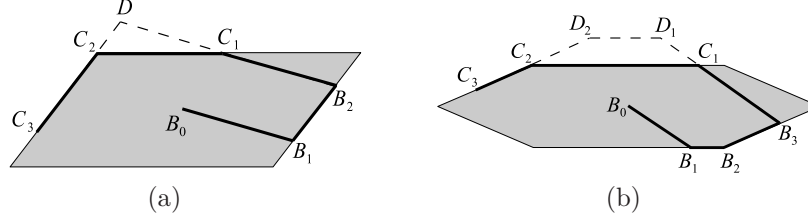


Figure 6: Violations of conditions of Theorem 1. The shaded area represents the polytope  $\Pi \cap V$ . The vector  $f_0$  used in the definition (24) of the Moreau process points in the direction of the vector  $B_0B_1$ . (a) The prism  $\Omega$  does not belong to  $\Pi \cap V$ . The polyline  $B_0B_1B_2C_1C_2C_3$  that contains the the polyline  $\gamma = B_0B_1B_2$  represents the trajectory of a solution  $u$  to (24) for the input  $g(t)f_0$  where  $g$  increases from zero to a maximum value  $g_*$  and then decreases to the minimum value  $-g_*$ . Formula (34) defines a different polyline  $B_0B_1B_2DC_3$  for the same input. (b) Conditions (29)–(30) are violated because  $\dim F_1 = \dim F_2$ . The polylines  $B_0B_1B_2B_3C_1C_2C_3$  and  $B_0B_1B_2B_3D_1D_2C_3$  the trajectory of the inclusion (24) and curve prescribed by formula (34) in response to the same input as in panel (a).

cases, we see that the trajectory of the Moreau process does not have the shape prescribed by formula (34), which generalizes the Prandtl-Ishlinskii operator.

4. The assumption that the map  $u^* : [0, L] \rightarrow \gamma$  is invertible is made for simplicity. It is straightforward to extend the theorem to the case when it is not satisfied.

5. Examples in which the system of springs shown in Fig. 1 cannot be reduced to a Prandtl-Ishlinskii operator can be found in [3].

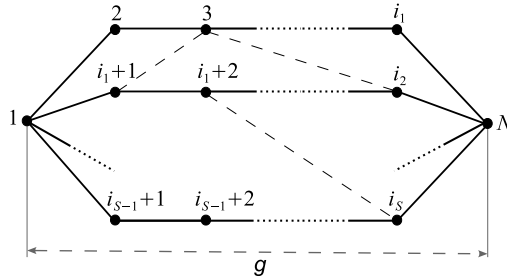


Figure 7: A graph  $G$  of a general *linear* connection of  $m$  springs (thick lines). Thin dashed lines represent possible additional edges as introduced in Theorem 2.

6. Examples with complex topology that satisfy the conditions of Theorem 1 can be created by a perturbation of simple systems considered in Section 2.6. In particular, let us consider a *linear* connection of springs with the corresponding graph  $\Gamma$  shown in Fig. 7 (the graph shown in Fig. 5 is an example of such a

graph). Here

$$1 = i_0 < i_1 < \dots < i_{S-1} < i_S = N - 1$$

and the list of all edges  $(ij)$  with  $i < j$  is as follows: node 1 is connected with nodes  $\{i_0+1, i_1+1, \dots, i_{S-1}+1\}$ ; node  $N$  is connected with nodes  $\{i_1, i_2, \dots, i_S\}$ ; and, each node  $i$  with  $i_n + 1 \leq i < i_{n+1}$  is connected with node  $i + 1$  for  $0 \leq n \leq S - 1$ . The total number of edges equals  $m = i_S + S - 1$ . Denote

$$\tilde{a}_n := \left( \frac{1}{a_{1, i_n+1}} + \sum_{i=i_n+1}^{i_{n+1}-1} \frac{1}{a_{i, i+1}} + \frac{1}{a_{i_{n+1}, N}} \right)^{-1}, \quad (47)$$

$$\tilde{r}_n := \min\{r_{1, i_n+1}, r_{i_n+1, i_n+2}, \dots, r_{i_{n+1}-1, i_{n+1}}, r_{i_{n+1}, N}\} \quad (48)$$

with  $n = 0, \dots, S - 1$  (see example in Section 2.6) and set

$$q_n(i) = \begin{cases} i_n + 1, & i = 1, \\ i + 1, & i_n \leq i \leq i_{n+1} - 1, \\ N, & i_{n+1}. \end{cases}$$

**Theorem 2.** *Consider a linear connection of  $m$  springs (see Fig. 7). Assume that*

$$r_{i, q_n(i)} \neq r_{j, q_n(j)}, \quad i \neq j; \quad i, j \in \{1, i_n + 1, i_n + 2, \dots, i_{n+1}\}, \quad (49)$$

$$\frac{\tilde{r}_{n_1}}{\tilde{a}_{n_1}} \neq \frac{\tilde{r}_{n_2}}{\tilde{a}_{n_2}}, \quad n_1 \neq n_2 \quad (50)$$

for  $0 \leq n, n_1, n_2 \leq S - 1$ . Let us extend this system with any set of additional connections of nodes by springs with thresholds  $r_{ij}$  and Young's moduli  $a_{ij}$ . Then, given any  $r_{th} > 0$  there is a  $\delta = \delta(r_{th}) > 0$  such that if for all the added connections  $r_{ij} \geq r_{th}$  and simultaneously  $a_{ij} < \delta$ , i.e. the Young's moduli  $a_{ij}$  of the added springs are sufficiently small, then the extended system of springs satisfies all the conditions of Theorem 1.

*Proof of Theorem 2.* First, we consider the linear connection of  $m$  springs. For this connection, the set of geometric and moving affine constraints (13) has the form

$$\varepsilon_{1, i_n+1} + \sum_{i=i_n+1}^{i_{n+1}-1} \varepsilon_{i, i+1} + \varepsilon_{i_{n+1}, N} = g, \quad n = 0, \dots, S - 1.$$

Therefore, the subspace  $W \ni \varepsilon = e + p$  of  $\mathbb{E} = \mathbb{R}^m$  is defined by the relations

$$\varepsilon_{1, i_n+1} + \sum_{i=i_n+1}^{i_{n+1}-1} \varepsilon_{i, i+1} + \varepsilon_{i_{n+1}, N} = 0, \quad n = 0, \dots, S - 1,$$

its orthogonal complement  $W^\perp \ni \sigma$  is given by

$$\sigma_{1, i_n+1} = \sigma_{i_n+1, i_n+2} = \dots = \sigma_{i_{n+1}-1, i_{n+1}} = \sigma_{i_{n+1}, N}, \quad n = 0, \dots, S - 1,$$

and the components  $k_0^{i,j}$  of the vector  $k_0$  in (13) have the form

$$k_0^{i,j} = \begin{cases} 1, & i = i_n, j = N, \\ 0, & \text{otherwise,} \end{cases} \quad n = 1, \dots, S.$$

Therefore, the orthogonal subspaces  $U = A^{\frac{1}{2}}W \ni u$  and  $V = A^{-\frac{1}{2}}W^\perp = U^\perp \ni v$  (see (17)) are defined by the systems

$$\frac{u_{1,i_n+1}}{\sqrt{a_{1,i_n+1}}} + \sum_{i=i_n+1}^{i_{n+1}-1} \frac{u_{i,i+1}}{\sqrt{a_{i,i+1}}} + \frac{u_{i_{n+1},N}}{\sqrt{a_{i_{n+1},N}}} = 0,$$

$$v_{1,i_n+1}\sqrt{a_{1,i_n+1}} = v_{i_n+1,i_n+2}\sqrt{a_{i_n+1,i_n+2}} = \dots = v_{i_{n+1},N}\sqrt{a_{i_{n+1},N}} \quad (51)$$

with  $n = 0, \dots, S-1$ , respectively. From these relations, it follows that the components of vector (19) are given by

$$f_0^{1,i_n+1}\sqrt{a_{1,i_n+1}} = f_0^{i_n+1,i_n+2}\sqrt{a_{i_n+1,i_n+2}} = \dots = f_0^{i_{n+1},N}\sqrt{a_{i_{n+1},N}} = \tilde{a}_n, \quad (52)$$

and the polytope  $\Pi \cap V$  is an  $S$ -dimensional parallelepiped defined by relations (51) and

$$|v_{1,i_n+1}|\sqrt{a_{1,i_n+1}} \leq \tilde{r}_n, \quad (53)$$

where  $\tilde{a}_n, \tilde{r}_n$  are defined in (59), (48). Using (49), (50), without loss of generality, we can assume that

$$\tilde{\rho}_0 < \tilde{\rho}_1 < \dots < \tilde{\rho}_{S-1}, \quad \tilde{\rho}_n := \frac{\tilde{r}_n}{\tilde{a}_n}, \quad (54)$$

and  $\tilde{r}_n = r_{1,i_n+1}$  for each  $n = 0, \dots, S-1$ , i.e.

$$\tilde{r}_n = r_{1,i_n+1} < \min\{r_{i_n+1,i_n+2}, \dots, r_{i_{n+1}-1,i_{n+1}}, r_{i_{n+1},N}\}, \quad (55)$$

(because the edges can always be labeled so as to ensure these relationships). Now, from (43), (51) and (53), it is easy to derive explicit expressions for the components of vectors  $f_1, \dots, f_S$  and coordinates of points  $B_1, \dots, B_S$ :<sup>8</sup>

$$f_j^{i,q_n(i)} = \begin{cases} 0, & i \in \{1, i_n+1, i_n+2, \dots, i_{n+1}\}, n < j, \\ f_0^{i,q_n(i)}, & i \in \{1, i_n+1, i_n+2, \dots, i_{n+1}\}, n \geq j, \end{cases} \quad (56)$$

$$B_j^{1,i_n+1}\sqrt{a_{1,i_n+1}} = B_j^{i_n+1,i_n+2}\sqrt{a_{i_n+1,i_n+2}} = \dots = B_j^{i_{n+1},N}\sqrt{a_{i_{n+1},N}} = \tilde{a}_n \min\{\tilde{\rho}_n, \tilde{\rho}_{j-1}\}$$

with  $n = 0, \dots, S-1$ .

These explicit formulas ensure that:

- (i1) The point  $B_j \in V$  belongs to the open face  $(\Pi_1 \cap \dots \cap \Pi_j)^\circ$  of the parallelepiped  $\Pi$ , i.e.

$$B_j \in (\Pi_1 \cap \dots \cap \Pi_j)^\circ,$$

where  $\Pi_j$  denotes the facet

$$\Pi_j = \{v \in \Pi : v_{1,i_{j-1}+1} = r_{1,i_{j-1}+1}/\sqrt{a_{1,i_{j-1}+1}}\}$$

of  $\Pi$  for  $j = 1, \dots, S$ .

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<sup>8</sup>One can also use explicit balance of forces to obtain the same result.

(i2)  $V \cap \Pi_1 \cap \cdots \cap \Pi_j = F_j$  and conditions (29), (30) are satisfied.

(i3) The subspace  $V$  and the hyperplanes

$$\Xi_j = \{v \in \mathbb{E} : v_{i_{j-1}+1} = 0\}, \quad j = 1, \dots, S,$$

which are parallel to the facets  $\Pi_j$  of  $\Pi$ , are in general linear position in the sense that

$$\dim(V \cap \Xi_1 \cap \cdots \cap \Xi_j) = \dim(V) - j = S - j, \quad j = 1, \dots, S.$$

This implies  $\overset{\circ}{E}_j = V \cap \Xi_1 \cap \cdots \cap \Xi_j$  in the notation of the previous section.

(i4)  $f_j = \text{proj}_{V \cap \Xi_0 \cap \Xi_1 \cap \cdots \cap \Xi_j} A^{\frac{1}{2}} k_0 \neq 0$  for  $j = 0, \dots, S-1$  (where  $\Xi_0 = \mathbb{E}$ ).

The next step is to show that the same properties (i)-(iv) are satisfied for the extended system of  $\tilde{m}$  springs. To this end, we first consider a simple embedding of the geometric picture considered above into the space  $\tilde{\mathbb{E}} = \mathbb{R}^{\tilde{m}}$  by identifying each vector  $v \in \mathbb{E} = \mathbb{R}^m$  with components  $v_{ij}$ ,  $(ij) \in M$ , with the vector  $\tilde{v} \in \tilde{\mathbb{E}}$  with the components

$$\tilde{v}_{ij} = \begin{cases} v_{ij}, & (ij) \in M, \\ 0, & (ij) \in \tilde{M} \setminus M. \end{cases}$$

(recall that  $\Gamma$  is a subgraph of  $\tilde{\Gamma}$ , hence  $M \subset \tilde{M}$ ). With this identification,  $\mathbb{E}$  becomes a subspace  $\tilde{\mathbb{E}}$  and we denote by  $\mathbb{E}^\perp$  its orthogonal complement with respect to the scalar product

$$\langle y, z \rangle = \sum_{(ij) \in \tilde{M}} y_{ij} z_{ij}, \quad y, z \in \tilde{\mathbb{E}},$$

(cf. (12)), i.e.  $\mathbb{E}^\perp = \{v \in \tilde{\mathbb{E}} : v_{ij} = 0, (ij) \in \tilde{M} \setminus M\}$ . Also, define the sets

$$\tilde{V} = V \oplus \mathbb{E}^\perp, \quad \tilde{\Pi} = \Pi \oplus \mathbb{E}^\perp, \quad \tilde{\Pi}_j = \Pi_j \oplus \mathbb{E}^\perp, \quad \tilde{\Xi}_j = \Xi_j \oplus \mathbb{E}^\perp$$

and the diagonal  $\tilde{m} \times \tilde{m}$  matrix  $\tilde{A}$  by

$$\tilde{A}v = \begin{cases} Av, & v \in \mathbb{E}, \\ 0, & v \in \mathbb{E}^\perp. \end{cases}$$

Then, it is easy to see that all statements (i1)-(i4), in which we replace the space  $\mathbb{E}$  with  $\tilde{\mathbb{E}}$ , the subspaces  $V, \Xi_j \subset \mathbb{E}$  with  $\tilde{V}, \tilde{\Xi}_j \subset \tilde{\mathbb{E}}$ , the parallelepipeds  $\Pi, \Pi \cap V$  and their faces  $\Pi_j, F_j$  with the polyhedra  $\tilde{\Pi}, \tilde{P} \cap \tilde{V}$  and their faces  $\tilde{P}_j, \tilde{F}_j$ , and the matrix  $A$  with the matrix  $\tilde{A}$ , remain valid. That is,

(j1)  $B_j \in \tilde{V} \cap (\tilde{\Pi}_1 \cap \cdots \cap \tilde{\Pi}_j)$  for  $j = 1, \dots, S$ .

(j2) The faces  $\tilde{F}_j = \tilde{V} \cap \tilde{\Pi}_1 \cap \cdots \cap \tilde{\Pi}_j$  of  $\tilde{\Pi} \cap \tilde{V}$  satisfy conditions (29), (30).

- (j3) The subspace  $\tilde{V}$  and the hyperplanes  $\tilde{\Xi}_j$ ,  $j = 1, \dots, S$ , are in general linear position.
- (j4) The vectors  $f_j$  satisfy  $f_j = \text{proj}_{\tilde{V} \cap \tilde{\Xi}_0 \cap \tilde{\Xi}_1 \cap \dots \cap \tilde{\Xi}_j} \tilde{A}^{\frac{1}{2}} k_0 \neq 0$  for  $j = 0, \dots, S-1$ .

Finally, consider the Moreaux process corresponding to the system of  $\tilde{m}$  springs, in which the springs with  $(ij) \in \tilde{M} \setminus M$  have small Young's moduli  $a_{ij}$ . For this system, we denote by  $\hat{k}_0, \hat{W}, \hat{V} \subset \tilde{\mathbb{E}}$  and  $\hat{A}$  the counterparts of the vector  $k_0$ , the subspaces  $W, V$  and the matrix  $A$  (cf. Section 2.4). Furthermore, we denote by  $\hat{\Pi}, \hat{\Pi}_j$  the parallelepiped

$$\hat{\Pi} = \{v \in \tilde{\mathbb{E}} : |v_{ij}| \leq r_{ij}/\sqrt{a_{ij}}, (ij) \in \tilde{M}\}$$

and its facets

$$\hat{\Pi}_j = \{v \in \hat{\Pi} : v_{1, i_{j-1}+1} = r_{1, i_{j-1}+1}/\sqrt{a_{1, i_{j-1}+1}}\}. \quad (57)$$

Note that the subspace  $\hat{V}$  is a small perturbation of the subspace  $V$  in the sense that the intersections of  $V$  and  $\hat{V}$  with any ball centered at the origin can be made arbitrarily close by making the coefficients  $a_{ij}$ ,  $(ij) \in \tilde{M} \setminus M$ , sufficiently small. Similarly, the sets  $\hat{\Pi}, \hat{\Pi}_j$  are small perturbations of  $\Pi, \Pi_j$ , respectively. Also, the matrix  $\hat{A}$  and hence the vector  $\hat{f}_0 = \hat{A}^{\frac{1}{2}} \hat{k}_0$  are small perturbations of the matrix  $\tilde{A}$  and the vector  $f_0 = \tilde{A}^{\frac{1}{2}} k_0$ , respectively, because  $\hat{k}_0^{i,j} = k_0^{i,j}$  for  $(ij) \in M$ . Therefore, properties (j3), (j4) imply that the subspace  $\hat{V}$  and the hyperplanes  $\tilde{\Xi}_j$ ,  $j = 1, \dots, S$ , are in general linear position; each subspace  $\hat{V} \cap \tilde{\Xi}_0 \cap \tilde{\Xi}_1 \cap \dots \cap \tilde{\Xi}_j$  is a small perturbation of the subspace  $\tilde{V} \cap \tilde{\Xi}_0 \cap \tilde{\Xi}_1 \cap \dots \cap \tilde{\Xi}_j$ ; and, each vector  $\hat{f}_j = \text{proj}_{\hat{V} \cap \tilde{\Xi}_0 \cap \tilde{\Xi}_1 \cap \dots \cap \tilde{\Xi}_j} \hat{A}^{\frac{1}{2}} \hat{k}_0$  is a small perturbation of the vector  $\tilde{f}_j \neq 0$ , hence  $\hat{f}_j \neq 0$ . Now, since  $B_j$  is the intersection point of the ray  $\{y = B_{j-1} + df_{j-1} : d > 0\}$  with the parallelepiped  $\hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_j$  for every  $j = 1, \dots, S$ , property (j1) implies that the relation

$$\psi_j(x) := \{y = x + df_{j-1} : d > 0\} \cap (\hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_j)$$

uniquely defines a point  $\psi_j(x) \in (\hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_j)$  whenever a point  $x \in \hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_{j-1}$  is sufficiently close to  $B_{j-1}$  and the vector  $\hat{f}_{j-1}$  is sufficiently close to  $f_{j-1}$ . Therefore, for sufficiently small  $a_{ij}$ ,  $(ij) \in \tilde{M} \setminus M$ , the points  $\hat{B}_0 = 0, \hat{B}_1 = \psi_1(\hat{B}_0), \hat{B}_2 = \psi_2(\hat{B}_1), \dots, \hat{B}_S = \psi_S(\hat{B}_{S-1})$  satisfy

$$\hat{B}_j \in (\hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_j), \quad \hat{B}_j = \hat{B}_{j-1} + (\hat{d}_j - \hat{d}_{j-1})\hat{f}_{j-1}, \quad j = 1, \dots, S, \quad (58)$$

with  $0 = \hat{d}_0 < \hat{d}_1 < \dots < \hat{d}_S$ , and the polyline  $\hat{\gamma} = \hat{B}_0 \hat{B}_1 \dots \hat{B}_S$  is a small perturbation of the polyline  $\gamma = B_0 B_1 \dots B_S$ . Since  $\hat{f}_j \in \hat{V}$  for all  $j$ , we also see that the vertex  $\hat{B}_j$  of  $\hat{\gamma}$  belongs to the open face  $\hat{F}_j = \hat{V} \cap (\hat{\Pi}_1 \cap \dots \cap \hat{\Pi}_j)$  of the polytope  $\hat{V} \cap \hat{\Pi}$  and the faces  $\hat{F}_j$  satisfy conditions (29), (30).

It remains to show that the set

$$\hat{\Omega} = \{y = \tau_1 \hat{B}_0 \hat{B}_1 + \cdots + \tau_S \hat{B}_{S-1} \hat{B}_S, |\tau_i| \leq 1, i = 1, \dots, S\} \subset \hat{V}$$

(cf. (31)) belongs to  $\hat{\Pi}$ . To this end, denote by  $e_{ij}$  the standard basis of vectors  $e_{i,j}$  in  $\tilde{\mathbb{E}}$ :

$$e_{i,j}^{i',j'} = \begin{cases} 1, & (i',j') = (ij), \\ 0, & \text{otherwise.} \end{cases}$$

For  $(i,j) = (1, i_n + 1)$ ,  $n = 0, \dots, S-1$ , consider the estimate

$$|\langle e_{1,i_n+1}, \sum_{j=1}^S \tau_j \hat{B}_{j-1} \hat{B}_j \rangle| \leq \sum_{j=1}^S |\langle e_{1,i_n+1}, \hat{B}_{j-1} \hat{B}_j \rangle|$$

with  $|\tau_j| \leq 1$ . Here  $\hat{B}_{n+1}, \dots, \hat{B}_S \in \hat{\Pi}_{n+1}$ , hence  $\langle e_{1,i_n+1}, B_{j-1} B_j \rangle = 0$  for  $j = n+2, \dots, S$  and therefore

$$|\langle e_{1,i_n+1}, \sum_{j=1}^S \tau_j \hat{B}_{j-1} \hat{B}_j \rangle| \leq \sum_{j=1}^{n+1} |\langle e_{1,i_n+1}, \hat{B}_{j-1} \hat{B}_j \rangle|. \quad (59)$$

But relations (52), (56) imply that  $\langle e_{1,i_n+1}, f_{j-1} \rangle > 0$  for  $1 \leq j \leq n+1 \leq S$  and since  $\hat{f}_{j-1}$  is a small perturbation of the vector  $f_{j-1}$ ,

$$\langle e_{1,i_n+1}, \hat{f}_{j-1} \rangle > 0, \quad 1 \leq j \leq n+1 \leq S.$$

Combining this estimate with the second relation in (58) results in the relation  $\langle e_{1,i_n+1}, \hat{B}_{j-1} \hat{B}_j \rangle > 0$  for  $j = 1, \dots, n+1$ , hence (59) is equivalent to

$$|\langle e_{1,i_n+1}, \sum_{j=1}^S \tau_j \hat{B}_{j-1} \hat{B}_j \rangle| \leq \sum_{j=1}^{n+1} \langle e_{1,i_n+1}, \hat{B}_{j-1} \hat{B}_j \rangle = \langle e_{1,i_n+1}, \hat{B}_0 \hat{B}_{n+1} \rangle = \frac{r_{1,i_n+1}}{\sqrt{a_{1,i_n+1}}}, \quad (60)$$

where the last equality follows from  $B_{n+1} \in \hat{\Pi}_{n+1}$  and (57).

Now, consider  $(ij) \neq (1, i_{n+1}) \in M$ , i.e.  $(i,j) = (i, q_n(i))$  for some  $0 \leq n \leq S-1$ ,  $i_n + 1 \leq i \leq i_{n+1}$ . Formulas (52), (56) imply that  $\langle e_{ii'}, f_{j-1} \rangle \geq 0$  for all  $(ii') \in M$ ,  $j = 1, \dots, S$  and  $\langle e_{i,q_n(i)}, f_{j-1} \rangle = 0$  for  $j > n+1$ . Therefore, taking into account that  $B_{j-1} B_j = (d_j - d_{j-1}) f_{j-1}$  and arguing exactly in the same way as above, we see that the unperturbed polyline  $\gamma$  satisfies

$$|\langle e_{i,q_n(i)}, \sum_{j=1}^S \tau_j B_{j-1} B_j \rangle| \leq \sum_{j=1}^{n+1} \langle e_{i,q_n(i)}, B_{j-1} B_j \rangle.$$

Furthermore, formulas (52), (56) and  $B_{j-1} B_j = (d_j - d_{j-1}) f_{j-1}$  imply

$$\langle e_{i,q_n(i)}, B_{j-1} B_j \rangle \sqrt{a_{i,q_n(i)}} = \langle e_{1,i_n+1}, B_{j-1} B_j \rangle \sqrt{a_{1,i_n+1}}, \quad j = 1, \dots, n+1,$$



hence

$$|\langle e_{i,q_n(i)}, \sum_{j=1}^S \tau_j B_{j-1} B_j \rangle| \leq \langle e_{1,i_n+1}, B_0 B_{n+1} \rangle \sqrt{\frac{a_{1,i_n+1}}{a_{i,q_n(i)}}},$$

where, due to  $B_{n+1} \in \hat{\Pi}_{n+1}$ ,

$$\langle e_{1,i_n+1}, B_0 B_{n+1} \rangle \sqrt{a_{1,i_n+1}} = r_{1,i_n+1}.$$

Therefore, (55) implies

$$|\langle e_{i,q_n(i)}, \sum_{j=1}^S \tau_j B_{j-1} B_j \rangle| < \frac{r_{i,q_n(i)}}{\sqrt{a_{i,q_n(i)}}}.$$

Since  $\hat{\gamma}$  is a small perturbation of  $\gamma$ , a similar estimate is true for  $\hat{\gamma}$ :

$$|\langle e_{i,q_n(i)}, \sum_{j=1}^S \tau_j \hat{B}_{j-1} \hat{B}_j \rangle| < \frac{r_{i,q_n(i)}}{\sqrt{a_{i,q_n(i)}}}, \quad i_n + 1 \leq i \leq i_{n+1}. \quad (61)$$

Finally, for  $ii') \in \tilde{M} \setminus M$ ,

$$|\langle e_{i,i'}, \sum_{j=1}^S \tau_j \hat{B}_{j-1} \hat{B}_j \rangle| < \frac{r_{i,i'}}{\sqrt{a_{i,i'}}}, \quad (62)$$

because  $a_{i,i'}$  is small. Combining (60), (61) and (62), we obtain  $\hat{\Omega} \subset \hat{\Pi}$ . This completes the proof of Theorem 2.

## 5 Conclusions

We considered a set of Prandtl's elastic-ideal plastic elements, which are arranged into a network and deform quasistatically when the distance between two nodes is varied according to a given law (input). As in the general setting of the model of Moreau, no *a priori* constraint was imposed on the topology of the network. We defined the loading curve  $\phi$  for the corresponding sweeping process as a graph of the solution corresponding to the zero initial condition and an increasing input. It was shown that if this curve satisfies simple geometric conditions, then the structure of hysteresis loops of the model in the space of stresses extended by one dimension representing the input is similar to the structure of hysteresis loops of the PI model. Furthermore, the relationship between the input and the varying stress of each Prandtl's element in the network is given by a PI operator  $\mathcal{P}_{\phi_i}$ , and the loading curve  $\phi_i$  of this operator is the corresponding projection of the loading curve  $\phi$  of the sweeping process. The question of how the geometric conditions of the main theorem can be related to the topology and parameters of the network in general remains open. However, we showed that these conditions are satisfied for any Moreau network obtained

by a small perturbation of a PI model. In other words, they are satisfied for any network with sufficiently small coupling. This is in line with the results from [3] obtained by a different approach based on the composition property of PI operators.

In this work, we considered networks of springs aligned along a straight line and used the distance between two nodes A and B as the input. However, the results can be easily generalized to networks of Prandtl's elements connecting nodes in a two- or three-dimensional space. It would be interesting to consider other types of scalar-valued inputs, for example, the force applied at the node A. In this case, the input  $Z(t)$  of the sweeping process is a set with changing size and shape and hence is more complicated than the inputs we considered. This will be the subject of future work.

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